

HIP – 1997 – 55 / TH
September 26, 1997

A model for multi-quark systems

A. M. Green¹ and P. Pennanen²

*Helsinki Institute of Physics and Department of Physics
P.O. Box 9, FIN-00014 University of Helsinki, Finland*

Abstract

As a step towards understanding multi-quark systems abundant in nature we construct a model that reproduces the binding energies of static four-quark systems. These energies have been calculated using SU(2) lattice gauge theory for a set of six different geometries representative of the general case. The model is based on ground and excited state two-body potentials and multi-quark interaction terms.

PACS numbers: 11.15.Ha, 12.38.Gc, 13.75.-n, 24.85.+p

1 Introduction

The significant progress in lattice QCD has so far been restricted to systems with only a few quarks – up to three in most cases. However, in particle and nuclear physics there is also considerable interest in few- and multi-*hadron* systems beginning with the possibility of bound $K\bar{K}$ states. This prompts one to ask what lattice QCD can say about these more complex quark systems. It seems unlikely that, in the foreseeable future, the techniques of lattice QCD can be developed sufficiently to tackle these problems directly. Therefore, we are reduced to constructing models that can explain the lattice results for the simplest multi-quark systems in a way that can be readily extended to more complicated cases. With this in mind, the Helsinki group in Refs. [1, 2, 3] has calculated the energies of four quarks on a lattice in various geometries – namely, four quarks at the corners of rectangles, tetrahedra and other geometries as shown in Fig. 1. This is taken to be our 'experimental' data, which is to be explained by some model. Hopefully, this set of geometries is general enough for the model to also explain the energies of geometries in between those actually considered. Our philosophy is that, if *any* geometry cannot be fitted, then the model fails, since then there is no reason to expect configurations not checked explicitly to be fitted.

¹E-mail: green@phcu.helsinki.fi

²E-mail: Petrus@hip.fi

Due to limitations of computing resources the 'experimental' data is not for full QCD. Instead of SU(3) we use SU(2), which saves about an order of magnitude in computer time. The less than 10% quantitative differences observed in the glueball spectrum and topological susceptibility of these theories suggest that the relevant qualitative features are preserved [4]. Secondly, since the quarks are fixed in some geometry, they are static. Therefore this discussion is most relevant to the heaviest quarks, in analogy to the static potentials used successfully to describe quarkonia. The static approximation is currently being partially removed by applying the techniques of Ref. [5] to a system of two B mesons with only the b quarks static. A third approximation is that no quark-pair creation is allowed the so-called quenched approximation. The effects from this approximation have been found to be of the same magnitude as the use of SU(2) instead of SU(3), apart from string breaking effects at distances larger than we simulate. We concentrate here on attempting to understand the best simulation data available, i.e. for a system with four static SU(2) quarks in the quenched approximation.

Any model that can be extended to multi-quark systems must presumably treat only the quark degrees of freedom explicitly – with the gluon degrees of freedom entering only implicitly. This is the same philosophy as used, with much success, for interacting multi-nucleon systems. In the latter, the meson fields generating the interactions lead to effective internucleon potentials, which take the form of two-nucleon potentials and, to a lesser extent, three- and four-nucleon potentials. However, it must be remembered that multi-quark and multi-nucleon systems are very different, since in the former the underlying gluon fields are strongly self-interacting. It is, therefore, not at all clear that *any* effective model defined in terms of only quark degrees of freedom will be successful. It is the purpose of this article to see to what extent such a model can be developed for the four-quark case. If this trial fails, then there is no point in expecting it to work in even more complex quark problems i.e. a successful model for four quarks is necessary but not sufficient before considering any extension to even more quarks.

In Section 2 the model is introduced. The results are given in Section 3 and concluding remarks presented in Section 4.

2 The model

The basis A, B, \dots of the model contains pairings of the quarks in all possible ways. A normalization matrix \mathbf{N} contains the overlaps of any two states, $N_{AA} = \langle A|A \rangle$, $N_{AB} = \langle A|B \rangle, \dots$, while a potential matrix \mathbf{V} for an interaction V is defined as $V_{AA} = \langle A|V|A \rangle$, $V_{AB} = \langle A|V|B \rangle, \dots$. The object is then to compare the eigenvalues

$$[\mathbf{V} - E(4)\mathbf{N}] = 0 \tag{1}$$

with the lattice results – the success or failure of the model being to what extent the two can be made to agree.

As we only consider SU(2), there is no distinction between the group properties of quarks (q) and antiquarks (\bar{q}). As shown in Fig. 1, four such quarks can then be paired in three different ways

$$A = (q_1 q_3)(q_2 q_4), \quad B = (q_1 q_2)(q_3 q_4) \quad \text{and} \quad C = (q_1 q_4)(q_2 q_3), \quad (2)$$

where each $(q_i q_j)$ is a colour singlet. These three basis states are not orthogonal to each other and have in the weak coupling limit the condition [6]

$$|A + B + C\rangle = 0. \quad (3)$$

Since $\langle A|A \rangle = \langle B|B \rangle = \langle C|C \rangle = 1$, we get in this limit the equalities $\langle A|B \rangle = \langle B|C \rangle = \langle A|C \rangle = -1/2$.

The form of the potential is motivated by perturbative one-gluon exchange i.e.

$$V_{ij} = -\frac{1}{3} \sum_{i \leq j} \tau_i \tau_j v_{ij}, \quad (4)$$

where, for the ground state, we use the Cornell form of parameterization of the static two-quark potential

$$v_{ij} = -e/r_{ij} + b_s r_{ij} + c \quad (5)$$

fitted to measured lattice values. Then $V_{AA} = v_{13} + v_{24}$ etc. and

$$V_{AB} = V_{BA} = \langle A|V|B \rangle = -\frac{1}{2} (v_{13} + v_{24} + v_{12} + v_{34} - v_{14} - v_{23}). \quad (6)$$

Here it should be noted that using this perturbative form of the potential without a multi-quark interaction term is unacceptable because of the resulting unphysical long range Van der Waals forces. When moving to stronger couplings the overlap between any two states should decrease, becoming zero in the strong coupling (or large distance) limit. We take this effect into account by introducing a factor f , defined as $\langle A|B \rangle = -f/2$, which decreases with increasing separation of the quarks.

For internal consistency, this same factor must also multiply V_{AB} in Eq. 6 – otherwise the eigenvalues would depend on the self-energy term c in Eq. 5. A reasonable parameterization for the multi-quark interaction term f is

$$f = \exp(-b_s k_f S), \quad (7)$$

where b_s is the string energy of Eq. 5, S the minimum area bounded by the straight lines connecting the quarks and k_f a parameter to be fitted.

This form of the parameterization was originally motivated by strong coupling ideas [7, 8]. More complicated parameterizations, such as $f = f_0 \exp(-b_s k_f S - \sqrt{b_s} k_P P)$ with P the perimeter of S and f_0, k_P new constants to be fitted, have been suggested. However, the new parameters were found to measure lattice artefacts in a continuum extrapolation, where $f_0 \rightarrow 1$, $k_P \rightarrow 0$ [9]. The determination of a minimal area S is a laborious task [10] and thus impractical when many such areas are needed. Therefore, we use a formula that is readily evaluated for any geometry; the area is simply taken to be the average of the sum of the four triangular areas defined by the positions of the four quarks i.e. the faces of the tetrahedron. For example, in the notation of Eq. 2, the appropriate area $S(AB)$ for f is

$$S(AB) = 0.5[S(431) + S(432) + S(123) + S(124)], \quad (8)$$

where $S(ijk)$ is the area of the triangle with corners at i, j and k . For planar geometries this simply reduces to the expected area, while for non-planar cases this is only an approximation to $S(AB)$.

Of the six geometries considered here the tetrahedron presents the most challenging data. Other geometries could be fitted qualitatively with the above model using only the one free parameter k_f in Eq. 7. However, the regular tetrahedron has maximal degeneracy as all quarks are at an equal distance from each other. Its energies have the interesting feature that the lowest state is doubly degenerate and becomes *more* bound as the tetrahedron increases in size, as noted in Ref. [11]. This is opposite to what happens with squares, where the magnitude of E_0 decreases with increasing size of the system. This indicates that there is coupling to some higher state(s) that becomes more effective as the size increases and suggests that these higher states contain gluon excitation with respect to the basic A, B, C configurations. Therefore, as proposed in Ref. [9], we include states, denoted with a prime, which describe the same quark partitions but with the gluonic field in an excited state. These excited states could be excitations of the two-body potential or flux configurations where the four quarks form a singlet. In the latter case the formalism below needs to be slightly modified. If the excited state is taken as the lowest two-body excitation, for which the gluonic field has the symmetry of the E_u representation of the lattice symmetry group D_{4h} , we have

$$A^* = (q_1 q_3)_{E_u} (q_2 q_4)_{E_u} \text{ etc.} \quad (9)$$

Because the E_u state is an odd parity excitation, A^*, B^*, C^* must contain two such states in order to have the same parity as A, B, C . The excitation energy of an E_u state over its ground state (A_{1g}) counterpart is $\approx \pi/R$ for two quarks a distance R apart. As said above, when R increases this excitation energy decreases making the effect of the A^*, B^*, C^* states more important.

In analogy to Eq. 3 an antisymmetry condition $A^* + B^* + C^* = 0$ holds for the excited states. As the gluonic excitations are orthogonal to the the ground state

we also have $\langle A|A^* \rangle = \langle B|B^* \rangle = \langle C|C^* \rangle = 0$. There are now two more functions $f^{a,c}$ defined as

$$\begin{aligned} \langle A^*|B^* \rangle &= \langle A^*|C^* \rangle = \langle B^*|C^* \rangle = -f^c/2 \quad \text{and} \\ \langle A^*|B \rangle &= \langle A^*|C \rangle = \dots \quad \text{etc} \quad \dots = -f^a/2. \end{aligned} \quad (10)$$

Here it is assumed that $f^{a,c}$ are both dependent on S as defined in Eq. 8. Since f^c involves only the excited states, it is reasonable to expect it has a form similar to f in Eq. 7 i.e.

$$f^c = \exp(-b_s k_c S). \quad (11)$$

In the weak coupling limit, from the $A^* + B^* + C^* = 0$ condition, we expect $\langle A|B^* \rangle = \langle B|C^* \rangle = \dots = 0$ at small distances. This also happens in the Isgur-Paton model, as can be seen in Fig. 6 of Ref. [9]. To take this into account we parameterize f^a as

$$f^a = (f_0^a + f_S^a S) \exp(-b_s k_a S). \quad (12)$$

Justification of this is given by the Isgur-Paton calculation and an analysis of the correlations between the paths involved in the simulation, which gives the term linear in S . When all three parameters are varied in a `Minuit` fit it is found that $f_0^a = 0.002(30)$ is consistent with zero as expected. Therefore, from now on we set this parameter to zero.

For the potential matrix $\mathbf{V}(\mathbf{f})$ the diagonal matrix elements, before the lowest energy amongst the basis states – usually V_{AA} – is removed, are

$$\langle A^*|V|A^* \rangle = v^*(13) + v^*(24), \quad \text{etc},$$

where $v^*(ij)$ is the potential of the E_u state – a quantity also measured on the lattice along with the four-quark energies. However, to allow more freedom in the following fits, we introduce a parameter b_0 in

$$\langle A^*|V|A^* \rangle = V_{AA} + b_0 V_{AA}^*; \quad V_{AA}^* = v^*(13) + v^*(24) - v(13) - v(24) \quad (13)$$

for state A^* and analogously for B^*, C^* .

If E_u is the most important excitation in the parameter fit, b_0 should turn out to be of the order unity. However, in practice we get 2.3(5), suggesting that excitations with higher energies are, perhaps, more relevant.

The two-quark potentials $v(ij)$ are taken to be more elaborate than the three term form of Eq. 5. They are fitted to the lattice data using [12]

$$v(r_{ij}) = 0.562 + 0.0696 r_{ij} - \frac{0.255}{r_{ij}} - \frac{0.045}{r_{ij}^2}. \quad (14)$$

Similarly, the excitation of the E_u state is fitted by

$$v^*(ij) - v(ij) = \frac{\pi}{r_{ij}} - \frac{4.24}{r_{ij}^2} + \frac{3.983}{r_{ij}^4}. \quad (15)$$

The extra terms containing r_{ij}^{-2} and r_{ij}^{-4} are purely for numerical reasons and ensure that the fitted values of $v(ij)$ and $v^*(ij)$ are, on average, well within 1% of the lattice values for all $r_{ij} \geq 2$.

There are two types of off-diagonal element

$$\langle A^*|V|B^* \rangle \quad \text{and} \quad \langle A|V|B^* \rangle. \quad (16)$$

These require further approximations. Using the Isgur-Paton model with $N=1$ as a guide we get *qualitatively*

$$\langle A^*|V|B^* \rangle = -\frac{f^c}{2} \left[V_{AA} + V_{BB} - V_{CC} + c_0 \frac{(V_{AA}^* + V_{BB}^*)}{2} \right] = \langle B^*|V|A^* \rangle \quad (17)$$

and analogously for the elements $\langle A^*|V|C^* \rangle, \langle B^*|V|C^* \rangle$.

Likewise,

$$\begin{aligned} \langle A|V|B^* \rangle &= -\frac{f^a}{2} \left[V_{AA} + V_{BB} - V_{CC} + a_0 \frac{V_{BB}^*}{2} \right] \\ \langle A^*|V|B \rangle &= -\frac{f^a}{2} \left[V_{AA} + V_{BB} - V_{CC} + a_0 \frac{V_{AA}^*}{2} \right] \quad \text{etc,} \end{aligned} \quad (18)$$

where a_0, c_0 are free parameters, which should have values of order unity if E_u is the most relevant excitation.

In the special case of regular tetrahedra, \mathbf{V} reduces to the form

$$\mathbf{V} = \left[\begin{array}{ccc|ccc} V_{AA} & -fV_{AA}/2 & -fV_{AA}/2 & 0 & -f^aV_a/2 & -f^aV_a/2 \\ -fV_{AA}/2 & V_{AA} & -fV_{AA}/2 & -f^aV_a/2 & 0 & -f^aV_a/2 \\ -fV_{AA}/2 & -fV_{AA}/2 & V_{AA} & -f^aV_a/2 & -f^aV_a/2 & 0 \\ \hline 0 & -f^aV_a/2 & -f^aV_a/2 & V_b & -f^cV_c/2 & -f^cV_c/2 \\ -f^aV_a/2 & 0 & -f^aV_a/2 & -f^cV_c/2 & V_b & -f^cV_c/2 \\ -f^aV_a/2 & -f^aV_a/2 & 0 & -f^cV_c/2 & -f^cV_c/2 & V_b \end{array} \right], \quad (19)$$

where $V_a = V_{AA} + a_0V_{AA}^*/2$, $V_b = V_{AA} + b_0V_{AA}^*$, $V_c = V_{AA} + c_0V_{AA}^*$. As with all geometries

$$\mathbf{N} = \left[\begin{array}{ccc|ccc} 1 & -f/2 & -f/2 & 0 & -f^a/2 & -f^a/2 \\ -f/2 & 1 & -f/2 & -f^a/2 & 0 & -f^a/2 \\ -f/2 & -f/2 & 1 & -f^a/2 & -f^a/2 & 0 \\ \hline 0 & -f^a/2 & -f^a/2 & 1 & -f^c/2 & -f^c/2 \\ -f^a/2 & 0 & -f^a/2 & -f^c/2 & 1 & -f^c/2 \\ -f^a/2 & -f^a/2 & 0 & -f^c/2 & -f^c/2 & 1 \end{array} \right]. \quad (20)$$

The full 6×6 matrix $[\mathbf{V} - E\mathbf{N}]$ now breaks into three 2×2 matrices, two of which are identical – giving the observed degeneracy. These have the form

$$[\mathbf{V} - E\mathbf{N}] = \begin{bmatrix} -E(1 + f/2) & -f^a(E - V_a)/2 \\ -f^a(E - V_a)/2 & -E(1 + f^c/2) + V_b + f^c V_c/2 \end{bmatrix} = 0 \quad (21)$$

whereas the third 2×2 matrix giving nondegenerate energies is

$$[\mathbf{V} - E\mathbf{N}] = \begin{bmatrix} -E(1 - f) & f^a(E - V_a) \\ f^a(E - V_a) & -E(1 - f^c) + V_b - f^c V_c \end{bmatrix} = 0. \quad (22)$$

3 Results

In Refs. [1, 2, 3, 11] four quark energies have been extracted for a variety of geometries using a $16^3 \times 32$ lattice with $\beta = 2.4$. This β value corresponds to a lattice spacing $a = 0.119(1)$ fm. From these energies, one hundred – distributed over all measured geometries – are selected for fitting. Configurations containing flux links of less than two lattice units were not included because of the strong lattice artefacts they contain. The six geometries are shown in Fig. 1. Specifically, we use 15 Tetrahedra (T), 6 Squares (S), 12 Rectangles (R) (including Tilted Rectangles (TR)), 4 Quadrilaterals (Q), 9 Non-Planar (NP) and 4 Linear (L). Only the lowest two energies ($E_{0,1}$) from the lattice simulation are used. In most cases a three basis simulation had been performed, so that a third energy (E_2) was in fact available. However, as this state is the highest calculated, it is not expected to be very reliable due to the higher excitations it contains. Its main purpose was to improve the estimate on E_1 by reducing its excited state contamination. One might question the connection of these energies with the continuum values. This has been answered in Ref. [9], where it was found that the binding energies for equal physical sizes essentially stay constant when the lattice spacing is made smaller.

Before commencing a fit, the size of the errors on the above data must be decided. The lattice simulation, through the boot-strap method, does indeed produce errors – statistical ones. However, some estimate must also be added for systematic errors. How this is done is somewhat subjective. Here the prescription is to assume all errors must be at least 0.005 and, also, at least 10% of the eigenvalue itself. The former corresponds to about 10%, 1% for the largest values of E_0, E_1 respectively.

The above 100 pieces of data were fitted with **Minuit** – the Migrad option – using the seven parameters: k_f in Eq. 7, f_S^a, k_a in Eq. 12, k_c in Eq. 11, b_0 in Eq. 13 and a_0, c_0 in Eqs. 18. The outcome yielded a $\chi^2/\text{d.o.f.}=1.08$ with the values of the parameters being given in Table 1.

Parameter	k_f	k_a	f_S^a	k_c	a_0	b_0	c_0
Value	1.25(6)	0.54(11)	0.046(3)	0.04(20)	4.4(3)	2.2(6)	8.0(4)

Table 1: The values of the parameters defining the interaction

Of these values:

- a) The observation that $k_c \approx 0$ implies that $f^c \approx 1$ i.e. the excited configurations interact amongst themselves in the way expected from perturbation theory.
- b) The values of b_0 (along with a_0, c_0) are somewhat larger than the naively expected value of unity. This suggests that higher order effects are important.

Table 2 shows the contributions to the total $\chi^2/\text{d.o.f.}$ from each of the 12 types of data – i.e. from E_0, E_1 for the six geometries (T, S, R, Q, NP, L). For comparison the 2×2 basis model (A, B) with $f = 1$ is also shown. This corresponds to the absence of multi-quark interaction. In spite of the frequent use in the literature of such models based only on two-body potentials this seems to be a very poor choice.

Basis	6×6		2×2	
Geometry	$\chi^2(E_0)$	$\chi^2(E_1)$	$\chi^2(E_0)$	$\chi^2(E_1)$
T	0.18	0.20	2.4	21.4
S	0.01	0.11	6.2	40.0
R	0.17	0.09	9.6	63.9
Q	0.09	0.03	0.03	0.25
NP	0.08	0.01	0.06	1.53
L	0.01	0.11	0.01	0.10

Table 2: The contributions of the two states (E_0, E_1) of each of the six geometries to the total $\chi^2/\text{d.o.f.}$ of 1.08 (6×6 basis) and 146 (2×2).

4 Discussion

The above model with 6 basis states fits well all simulated geometries and confirms our earlier work that a 2×2 basis model with only two-quark interactions (i.e. $f = 1$) is not able to even qualitatively account for the data. Areas of further study include the actual nature of the higher excitations, which seem to be playing an important role. A natural extension of the present static quark model is the

application of the model in a dynamic case using the Schrödinger equation. Work on simulating such a system is in progress.

5 Acknowledgement

We thank J. Lukkarinen for sharing his L^AT_EX art. Funding from the Finnish Academy and M. Ehrnrooth foundation (P.P) is gratefully acknowledged. Our simulations were performed at the Center for Scientific Computing in Espoo, Finland.

References

- [1] A. M. Green, C. Michael and J. E. Paton, Nucl. Phys. **A 554**, 701 (1993).
- [2] A. M. Green, C. Michael, J. E. Paton and M. Sainio, Int. J. Mod. Phys. **E 2**, 479 (1993).
- [3] A. M. Green, C. Michael and J. E. Paton, Zeit. Phys. **C 67**, 291 (1995).
- [4] M. Teper, Phys. Lett. **B397**, 223 (1997).
- [5] C. Michael and J. Peisa, *Exotic hadronic states and all to all quark propagators*, preprint, appears on hep-lat/9705013 (1997).
- [6] A. M. Green and J. Paton, Nucl. Phys. **A492**, 595 (1989).
- [7] O. Morimatsu, Nucl. Phys. **A505**, 655 (1989).
- [8] C. Alexandrou, T. Karapiperis and O. Morimatsu, Nucl. Phys. **A 518**, 723 (1990).
- [9] P. Pennanen, Phys. Rev. **D 55**, 3958 (1997).
- [10] S. Furui and B. Masud, *An analysis of Four-quark Energies in SU(2) Lattice Monte Carlo for the Tetrahedral geometry*, in preparation.
- [11] A. M. Green, J. Lukkarinen, P. Pennanen and C. Michael, Phys. Rev. **D 53**, 261 (1996).
- [12] A. M. Green, C. Michael and P. S. Spencer, Phys. Rev. **D 55**, 1216 (1997).

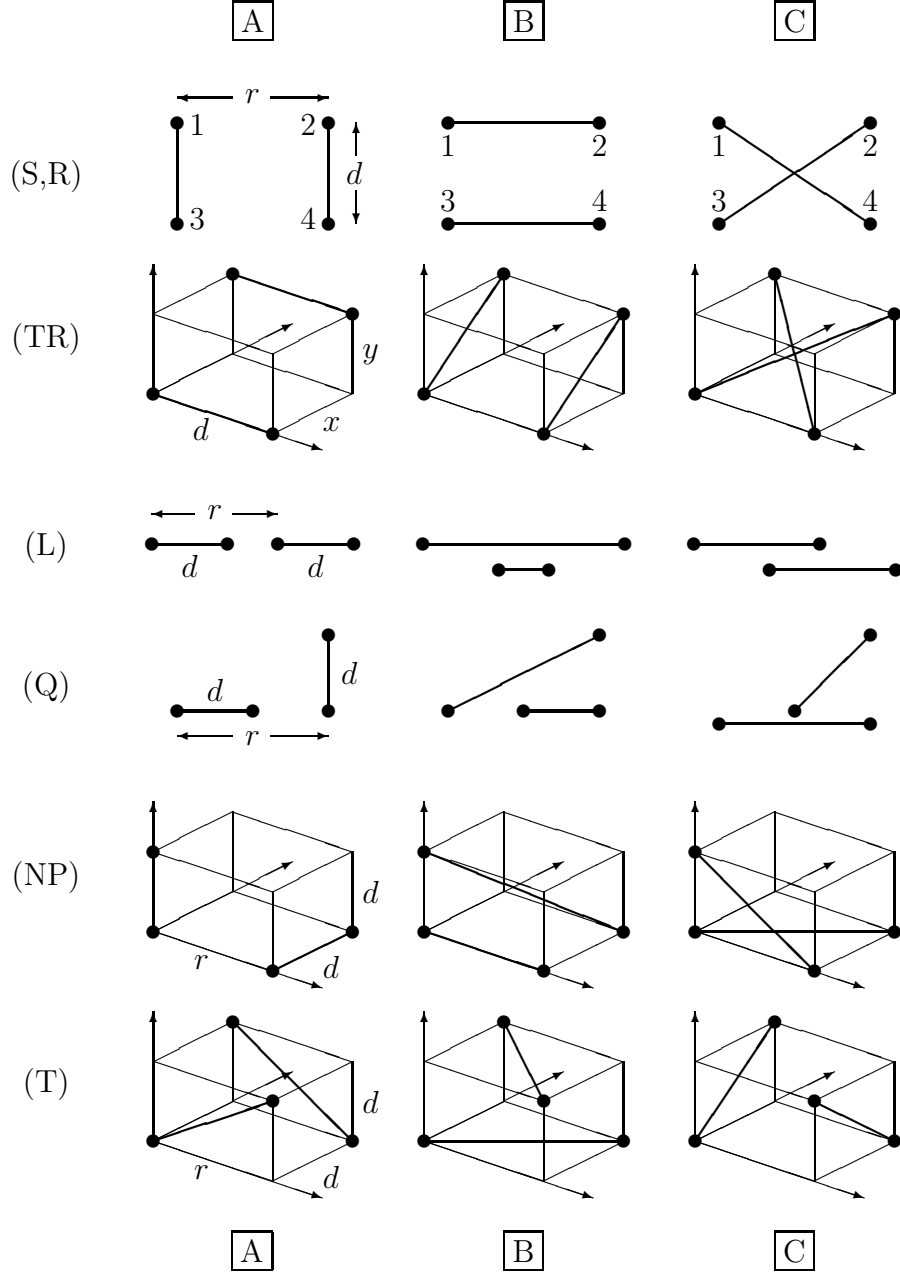


Figure 1: Simulated four-quark geometries and their two-body pairings.